XXII. *Consideration of various Points of Analysis.* *By* John F. W. Herschel, *Esq. F. R. S.*

Read May 19, 1814.

On the Calculus of Generating Functions.

In whatever point of view we consider the theory of generating functions, whether as the fertile source of new discoveries, or as a medium for exhibiting, in the most comprehensive and uniform point of view, results already known; we shall find fresh cause to admire the profound and original genius of its author. To the latter of these objects it is, however, more peculiarly adapted, and perhaps, in the present state of analytics, this may even be considered as the more precious advantage of the two. Such has been the indefatigable activity of those illustrious men, who have devoted themselves to the pursuit of mathematical science, that analysis must be considered as already adequate to every purpose to which we can reasonably hope to see it applied. The attention of the scientific observer must now be directed to those elevated stations, from which distinct and extended views of its arrangement as a whole can be obtained. The calculus of generating functions affords such a station, and commands a wider and more magnificent prospect than any which has yet been opened to the view of the speculative philosopher. It becomes interesting then to extend its application as far as possible in this line, and to introduce it on every occasion where there seems any

probability of its coming successfully into play. Such have been in part the considerations which determined me to adopt it as a vehicle in laying before the Society some results of a singular and interesting nature, derived indeed originally from other principles, but which, like all the rest, flow with the utmost facility from the first elements of this calculus.

In the following pages I have uniformly made use of the functional or characteristic notation; together with the method of separating (where it could conveniently be done) the symbols of operation from those of quantity. This method I have, perhaps, extended and carried somewhat farther than has hitherto been customary; but, I trust, without losing sight of its grand and ultimate object, the union of extreme generality with conciseness of expression. To avoid the necessity of continual explanation, I shall here set down the leading points of the system.

I. The signs : $\times$ ( ) are used to separate the symbol of operation from that of the quantity operated upon, thus:

$$f(x),\ \varphi : \log x,\ \left\{ e^{\frac{d}{dx}} - 1 \right\}^n \times \varphi(x).$$

II. 1. The combination of two operations is represented by placing their symbols together in their proper order. Thus, $\varphi(\psi(x))$ is simply written $\varphi\psi(x)$. For example, if $\varphi(x) = 1 + x$, and $\psi(x) = x^2$, then $\varphi\psi(x) = 1 + x^2$, and $\psi\varphi(x) = (1 + x)^2$.

2. When several operations combined are considered as one, their characteristics are inclosed in parentheses ( ). Thus

$$f \log \varphi(x) = (f \log \varphi) : x.$$

3. The repetition of the same operation f being denoted (by the first rule of this article) by ff, fff, &c. may be more con-

cisely represented thus, $f^2, f^3, \ldots . f^n$; and this furnishes us with a general and very simple notation for the reverse operation of that denoted by f. For since $f^m f^n(x) = f^{m+n}(x)$, if we make $m = -1$, $n = +1$, we find $f^{-1} f(x) = f^0(x) = x$ with the operation f *no times* performed, $= x$. Thus f^{-1} is the characteristic of that operation which must be performed on $f(x)$ to reduce it to x: that is, of the reverse operation. This is surely a simpler and more expressive method than that of inverting the characteristic,* accentuating it on the left side,† or below;‡ or other similar contrivances. For instance,

$$\log^{-1} x = \varepsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c.$$

$$\tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$$

4. If a combination of operations, as $\varphi\psi$ &c. be n times repeated in their order, thus; $\varphi\psi\varphi\psi \ldots . (x)$, it will, by the second and third rules of this article be denoted by $(\varphi\psi)^n : x$. It must be observed, however, that $(\varphi\psi)^n$ is not the same as $\varphi^n \psi^n$, except in some particular cases.

III. 1. If any number of functions of a symbol x be added, subtracted, or otherwise combined, the resulting function is expressed by the same combination of their characteristics, observing the following conditions.

2. The actual multiplication of two functions $\varphi(x)$ and $\psi(x)$ is expressed by inserting a full point between their characteristics, thus, $\varphi . \psi(x) = \varphi(x) . \psi(x)$.

3. The actual elevation of a function to any power is thus

* LAPLACE. Journal de l'Ecole Polytechnique. No. 15.

† MONGE. Savans Etrangers. 1773.

‡ KNIGHT. Philosophical Transactions, 1811. Part I.

expressed, $\{f(x)\}^n = \{f\}^n : x$, to distinguish it from $f^n(x)$, whose signification has already been explained.

4. If F (x) be developable in any series of the form

$$ax^\alpha + bx^\beta + cx^\gamma + \&c.$$

the following abbreviations are used

$$(\mathrm{F} : \varphi) : x = a \,.\, \varphi^\alpha(x) + b \,.\, \varphi^\beta(x) + c \,.\, \varphi^\gamma(x) + \&c.$$

and, $(\mathrm{F}\{\varphi\}) : x = a \,.\, \{\varphi(x)\}^\alpha + b \,.\, \{\varphi(x)\}^\beta + \&c.$

Thus (for example)

$$\frac{1}{1 - \frac{d}{dx}} \times y = y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \&c.$$

$$\frac{1}{1 - \{\log.^{-1}\}} \times x = x + \varepsilon^x + \varepsilon^{2x} + \&c.$$

IV. 1. D is used as the sign of derivation. It is, properly speaking, the sign of an operation performed, not on quantity, but on the characteristic which it immediately precedes; by which the operation denoted by that characteristic is altered. For instance: D sin. = cos.; D cos. = − sin. But it must be observed that D $\log.^{-1} = \log.^{-1}$.

2. The sign D affects only the characteristic next following it, thus, $\mathrm{D}\varphi f(x) = (\mathrm{D}\varphi) : f(x)$. If it be intended to affect a combination of operations, the rule II, 2 must be observed. Thus, $\mathrm{D}(\varphi f) : x$, $\mathrm{D}^n(\psi \log.^{-1}) : \log. x$.

V. Every functional characteristic is affected by all the characteristics preceding it, in the same manner as if it were a symbol of quantity.

VI. Every characteristic of operation *performed on quantity* affects all which follows it, as if it were one symbol. Thus if $f(x) = ax^\alpha + bx^\beta + \&c.$; we shall have

$$f\mathrm{D}\varphi(x) = a \cdot \left\{\mathrm{D}\varphi(x)\right\}^{\alpha} + b \cdot \left\{\mathrm{D}\varphi(x)\right\}^{\beta} + \&c.$$

This rule does not extend to the signs D, d, Δ, δ, $\int$, Σ, according to the remark in IV. 1.

These rules will suffice to explain whatever may appear obscure or capricious in the following sheets. We shall now proceed to their practical application.

If $\varphi(t)$ be a function of t, developable in a series

$$\mathrm{A}_{-\infty} t^{-\infty} + \ldots\ldots \mathrm{A}_0 + \mathrm{A}_1 t + \ldots\ldots \mathrm{A}_x t^x + \ldots \mathrm{A}_{\infty} t^{\infty}$$

$\varphi(t)$ is said to be the generating function of A_x, and it may be said to be taken with respect to t. To this we shall appropriate a peculiar symbol G_t, as follows:

$$\varphi(t) = \mathrm{G}_t \left\{\mathrm{A}_x\right\}$$

When only one symbol t is used, the index below the G may be understood, and our equation will be

$$\varphi(t) = \mathrm{G}\left\{\mathrm{A}_x\right\}.$$

If $\varphi(t, t')$ be a function of t, t', developable in a double series

$$\begin{array}{l}
\mathrm{A}_{-\infty,-\infty} t^{\infty} \cdot t'^{-\infty} + \ldots \mathrm{A}_{-\infty,y} t^{-\infty} \cdot t'^{y} + \ldots \mathrm{A}_{-\infty,\infty} t^{-\infty} \cdot t'^{\infty} \\
+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
+ \mathrm{A}_{x,-\infty} t^{x} \cdot t'^{-\infty} + \ldots\ldots \mathrm{A}_{x,y} t^{x} \cdot t'^{y} + \ldots\ldots \mathrm{A}_{x,\infty} t^{x} \cdot t'^{\infty} \\
+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
+ \mathrm{A}_{\infty,-\infty} t^{\infty} \cdot t'^{-\infty} + \ldots\ldots \mathrm{A}_{\infty,y} t^{\infty} \cdot t'^{y} + \ldots\ldots \mathrm{A}_{\infty,\infty} t^{\infty} \cdot t'^{\infty}.
\end{array}$$

$\varphi(t, t')$ is said to be the generating function of $\mathrm{A}_{x,y}$ with respect to t, t', and may be thus expressed;

$$\varphi(t, t') = \mathrm{G}^2_{t,t'} \left\{\mathrm{A}_{x,y}\right\}$$

and so on, if there be any number of symbols t, t', t'', &c To

denote the sum of all the terms of a series, we shall use the sign S, thus,

$\varphi(t) = \mathrm{S}\left\{\mathrm{A}_x t^x\right\}$, and in like manner $\varphi(t, t') = \mathrm{S}^2_{x,y}\left\{\mathrm{A}_{x,y} \,.\, t^x \,.\, t'^y\right\}$.

For t let ht be written, and we obtain

$$\varphi(ht) = \mathrm{S}\left\{\mathrm{A}_x\ (ht)^x\right\} = \mathrm{S}\left\{\mathrm{A}_x h^x \,.\, t^x\right\} = \mathrm{G}\left\{\mathrm{A}_x h^x\right\}$$

Thus, if the generating function of A_x be $\varphi(t)$, that of $\mathrm{A}_x h^x$ will be $\varphi(ht)$.

Let this equation be multiplied by t^{-r}, and we get

$$t^{-r}\,\varphi(ht) = \mathrm{S}\left\{\mathrm{A}_x\, h^x \,.\, t^{x-r}\right\}$$
$$= \mathrm{S}\left\{\mathrm{A}_{x+r}\, h^{x+r} \,.\, t^x\right\} = \mathrm{G}\left\{\mathrm{A}_{x+r} \,.\, h^{x+r}\right\}.$$

If then the generating function of A_x be $\varphi(t)$, that of $\mathrm{A}_{x+r}\, h^{x+r}$ will be $t^{-r} \,.\, \varphi(ht)$.

Again, it is easy to see that

$$a \,.\, \mathrm{G}\left\{\mathrm{A}_x\right\} + b \,.\, \mathrm{G}\left\{\mathrm{B}_x\right\} + \text{\&c.} = \mathrm{G}\left\{a\mathrm{A}_x + b\mathrm{B}_x + \text{\&c.}\right\}$$

and thus we have

$$(at^{-\alpha} + bt^{-\beta} + ct^{-\gamma} + \text{\&c.}) \,.\, \varphi(ht) = \mathrm{G}\left\{a\mathrm{A}_{x+\alpha}\, h^{x+\alpha} + b\mathrm{A}_{x+\beta}\, h^{x+\beta} + \text{\&c.}\right\}$$

and, if $h = 1$

$$(at^{-\alpha} + bt^{-\beta} + \text{\&c.}) \,.\, \varphi(t) = \mathrm{G}\left\{a\mathrm{A}_{x+\alpha} + b\mathrm{A}_{x+\beta} + \text{\&c}\right\};(1).$$

Let us express the function $a\mathrm{A}_{x+\alpha} + b\mathrm{A}_{x+\beta} + \text{\&c}$ by the symbol $\nabla\mathrm{A}_x$ and let

$$at^{\alpha} + bt^{\beta} + ct^{\gamma} + \text{\&c.} = f(t)$$

$\nabla\mathrm{A}_x$ then will be the same as $f(\mathrm{A}_x)$, provided that in the

developement of $f(A_x)$, A_{x+i} be every where written for $\{A_x\}^i$, and to derive the generating function of ∇A_x from that of A_x, we have only to multiply the latter by $f\left(\frac{1}{t}\right)$, thus

$$G\left\{\nabla A_x\right\} = \varphi(t) \cdot f\left(\frac{1}{t}\right); \quad \ldots (2)$$

Let now $\nabla^2 A_x$, $\nabla^3 A_x$, &c. denote the respective values of the expression

$$aA_{x+\alpha} + bA_{x+\beta} + \&c.$$

When instead of A_x we write ∇A_x, $\nabla^2 A_x$, &c. that is,

$$\nabla^{i+1} A_x = a\nabla^i A_{x+\alpha} + b\nabla^i A_{x+\beta} + \&c.$$

and it is easy to see that we shall have

$$G\left\{\nabla^i A_x\right\} = \left\{f\left(\frac{1}{t}\right)\right\}^i \cdot \varphi(t); \quad \ldots \ldots (3).$$

Let us now denote by $'f\left(\frac{1}{t}\right)$ the expression

$$'a \,.\, t^{-'\alpha} + 'b \,.\, t^{-'\beta} + \&c.$$

and by $'\nabla^i A_x$ the function formed from $'f\left(\frac{1}{t}\right)$ in the same manner as we formed $\nabla^i A_x$ from $f\left(\frac{1}{t}\right)$. The equation (3) then may be thus written

$$G\left\{\nabla^i A_x\right\} = \left\{(f'f^{-1}) : 'f\left(\frac{1}{t}\right)\right\}^i \cdot \varphi(t)$$

If then $\left\{f'f^{-1}(t)\right\}^i$ be developed by the ordinary methods into a series of the form

$$a_{-\infty} t^{-\infty} + \ldots\ldots a_z t^z + \ldots\ldots a_\infty t^\infty = S_z\left\{a_z t^z\right\}$$

we shall obtain

$$G\left\{\nabla^i A_x\right\} = S_z\left\{a_z \,.\left\{'f\left(\frac{1}{t}\right)\right\}^z\right\}.\,\varphi(t) = S_z\left\{a_z \,.\, G\left\{'\nabla^z A_x\right\}\right\},$$

and of course

$$\overset{i}{\nabla} A_x = a_{-\infty}\, '\nabla^{-\infty} A_x + \ldots\ldots a_0 A_x + a_1\, '\nabla A_x + \ldots\ldots a_\infty\, '\nabla^\infty A_x; \ldots\ldots (4).$$

Thus we may always develope $\nabla^i A_x$ in a series containing only the successive orders of $'\nabla A_x$, such as $'\nabla^2 A_x$, &c.

If the developement of $\left\{f'f^{-1}(t)\right\}^i$ contain no negative powers of t, we have

$$a_z = \frac{D^z \left\{f'f^{-1}\right\}^i : o}{1.2 \ldots\ldots z}$$

and consequently

$$\nabla^i A_x = \left\{f'f^{-1}\right\}^i : o \,.\, A_x + \frac{D\left\{f'f^{-1}\right\}^i : o}{1} \,.\, '\nabla A_x + \frac{D^2\left\{f'f^{-1}\right\}^i : o}{1.2}\, '\nabla^2 A_x + \&c. \ldots\ldots (5).$$

Let $'\nabla A_x = A_{x+1} - A_x$, and we have $'f\left(\frac{1}{t}\right) = \frac{1}{t} - 1$, and $'f^{-1}(t) = 1 + t$, whence we obtain

$$\overset{i}{\nabla} A_x = \left\{f(1)\right\}^i . A_x + \frac{D\left\{f\right\}^i : 1}{1} \Delta A_x + \frac{D^2\left\{f\right\}^i : 1}{1.2} . \Delta^2 A_x + \&c.; \ldots\ldots\ldots (6)$$

for it is evident that when $t = o$, $D^z \left\{f(1+t)\right\}^i$ becomes $D^z \left\{f\right\}^i : 1$. To take a particular case, let $\nabla A_x = A_{x+1}$ and $\nabla^i A_x = A_{x+i}$, $f\left(\frac{1}{t}\right) = \frac{1}{t}$, and $D^z \left\{f\right\}^i : 1 = i\,(i-1) \ldots (i - z + 1)$, whence

$$A_{x+i} = A_x + \frac{i}{1} . \Delta A_x + \frac{i\,(i-1)}{1.2} . \Delta^2 A_x + \&c. \quad (7).$$

Again, if we suppose $\nabla A_x = \Delta A_x = A_{x+1} - A_x$, and $'\nabla A_x = A_{x+1}$, we shall obtain from (5)

$$\Delta^i A_x = A_{x+i} - \frac{i}{1} . A_{x+i-1} + \frac{i(i-1)}{1.2} . A_{x+i-2} - \&c; \ldots (8)$$

But to proceed. We have,

$$G\left\{x^i A_x\right\} = S\left\{A_x x^i . t^x\right\} = \frac{t}{dt} d \frac{t}{dt} d \ldots\ldots \frac{t}{dt} . d\varphi(t)$$

or which is the same thing,

$$G\left\{x^i A_x\right\} = \frac{1}{d\log t} d \frac{1}{d\log t} d \ldots\ldots\ldots d\left\{\varphi \log^{-1} : \log t\right\}$$

Now,

$$\frac{d\left\{\varphi \log^{-1} : \log t\right\}}{d\log t} = \frac{D(\varphi \log^{-1}) : \log t \times d\log t}{d\log t} = (\varphi\log^{-1}):\log t$$

$$\frac{d\left\{D(\varphi \log^{-1}) : \log t\right\}}{d\log t} = D^2(\varphi \log^{-1}) : \log t$$

and so on. Thus our equation becomes

$$G\left\{x^i A_x .\right\} = D^i(\varphi \log^{-1}) : \log t$$

and, if $f(x) = ax^\alpha + bx^\beta + \&c$, we see that

$$G\left\{A_x . f(x)\right\} = (aD^\alpha + bD^\beta + \&c)(\varphi \log^{-1}) : \log t$$

$$= (f : D)(\varphi \log^{-1}) : \log t ; \ldots\ldots (9)$$

If $f(x)$ be a rational integral function of x, the second member of this equation will require only the ordinary rules of the differential calculus for its formation, and of course the first may be rigorously obtained.

Conceive $f(t)$ and $F(t')$ to be developed into the two series $S_x\left\{a_x t^x\right\}$ and $S_y\left\{A_y t'^y\right\}$, and let us consider the double series

$$\sigma = S^2_{x,y}\left\{a_x A_y . h^{xy} . t^x . t'^y\right\}$$

First, σ may be expressed as follows,

$$\sigma = S_x\left\{a_x t^x . S_y\left\{A_y . (h^x t')^y\right\}\right\} = S_x\left\{a_x t^x . F(t' h^x)\right\}$$

But by a similar mode of reasoning, we should also find

$$\sigma = \mathrm{S}_y \left\{ \mathrm{A}_y t'^y . f(th^y) \right\}$$

or, $$\sigma = \mathrm{S}_x \left\{ \mathrm{A}_x t'^x . f(th^x) \right\}$$

for since x and y vary through all their values, these two sums are identical. Equating then the values of σ

$$\mathrm{S} \left\{ \mathrm{A}_x t'^x . f(th^x) \right\} = \mathrm{G}_t \left\{ a_x . \mathrm{F}(t' h^x) \right\}$$
$$= (\mathrm{F} : t' h^{\mathrm{D}}) (f \log^{-1}) : \log t$$

Let $t' = 1$, and for h writing h^{-1}, and adding or subtracting

$$\mathrm{S}\left\{ \mathrm{A}_x . \frac{f(th^x) \pm f(th^{-x})}{2} \right\} = \frac{\mathrm{F}(h^{\mathrm{D}}) \pm \mathrm{F}(h^{-\mathrm{D}})}{2} (f \log^{-1}) : \log t; \quad (10)$$

II. *On Logarithmic Transcendents.*

The equation we have just arrived at affords us a method of exhibiting, in a finite form, the sum of the series in its first member, provided we possess the means of obtaining the second; and it appears, by what we have before remarked, that this can be performed, whenever $\mathrm{F}(h^{\mathrm{D}}) \pm \mathrm{F}(h^{-\mathrm{D}})$ is a rational integral function of D. This includes among the forms of F those remarkable functions denominated by Mr. SPENCE, " Logarithmic Transcendents," and we shall now proceed, by the help of the general property demonstrated by that author in his " Essay &c." to derive from these principles the summation of one of the most extensive classes of series which has yet received discussion.

Adopting Mr. SPENCE's notation, we will represent the series

$$\frac{x}{1^n} - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \&c.$$

by the symbol ${}^n\mathrm{L}(1 + x)$. The property then alluded to is as follows:

$$\frac{{}^{0}\mathrm{L}\,(1+x)+(-1)^n.\,{}^{n}\mathrm{L}\,(1+x^{-1})}{2}=\frac{{}^{0}\mathrm{L}\,(2)\,.\,(\log.\,x)^n}{1.2\,....\,n}+\frac{{}^{2}\mathrm{L}\,(2)\,.\,(\log.\,x)^{n-2}}{1.2\,...\,(n-2)}+\&c.$$

continued as far as it will go without involving negative powers of log. x. Supposing then $\mathrm{F}\,(t)={}^{n}\mathrm{L}\,(1+t)$, and writing ε^{θ} for h we obtain

$$\mathrm{S}\left\{\frac{(-(1^{x+1}}{x^n}\,.\,\frac{f\,(t\varepsilon^{x\theta})+(-1)^n.\,f\,(t\varepsilon^{-x\theta})}{2}\right\}=$$

$$=\left(\frac{{}^{0}\mathrm{L}\,(2)\,.\,\theta^n}{1.2\,....\,n}\,\mathrm{D}^n+\frac{{}^{2}\mathrm{L}\,(2)\,.\,\theta^{n-2}}{1.2\,....\,(n-2)}\,\mathrm{D}^{n-2}+\&c.\right)(f\log^{-1}):\log t;\quad (11)$$

A very remarkable case of this equation is when $t=1$, or $\log t=o$, for then $\frac{\mathrm{D}^i\,(f\,\log^{-1}):\,\log t}{1.2\,...\,i}=$ the coefficient of t^i in the developement of f $\log^{-1}$ (t) or of $f\,(\varepsilon^t)$. If then we suppose

$$f\,(\varepsilon^t)=a_0+a_1\,t+a_2\,t^2+\,....\,a_n\,t^n+\&c.$$

we shall have the following equation:

$$\mathrm{S}\left\{\frac{(-1)^{x+1}}{x^n}\,.\,\frac{f\,(\varepsilon^{x\theta})+(-1)^n.\,f\,(\varepsilon^{-x\theta})}{2}\right\}={}^{0}\mathrm{L}\,(2)\,.\,a_n\,\theta^n+{}^{2}\mathrm{L}\,(2)\,.\,a_{n-2}\,\theta^{n-2}+\&c.;\ ($$

The second member being continued so long as it does not involve negative powers of θ.

With regard to the functions ${}^{0}\mathrm{L}(2)$, ${}^{2}\mathrm{L}(2)$, &c. we have, as is well known

$${}^{0}\mathrm{L}(2)=1-1+1-1+\&c.=\tfrac{1}{2}\text{, and}$$

$${}^{2x}\mathrm{L}(2)=\frac{(2^{2x-1}-1)\,\pi^{2x}}{1.2\,......\,(2x)}.\,\mathrm{B}_{2x-1}$$

B_{2x-1} being the xth number of BERNOUILLI.

The equation (12) by assigning specific forms to $f\,(t)$ affords an indefinite variety of interesting results, of which we shall only notice a few, the most remarkable.

1. Let $f(t)=\frac{-1}{\sqrt{-1}+t}$. and for n write $2n-1$, and $\theta\sqrt{-1}$ for θ.

The usual exponential expression for tan. θ, reduced into two fractions, gives

$$\tan.\ \theta = \frac{-1}{\sqrt{-1} + \varepsilon^{\theta\sqrt{-1}}} - \frac{-1}{\sqrt{-1} + \varepsilon^{-\theta\sqrt{-1}}}; \ldots\ldots\ldots\ldots (13)$$

Thus, the first member of (12) becomes

$$\tfrac{1}{2}\,\mathrm{S}\left\{(-1)^{x+1} \cdot \frac{\tan.\ x\theta}{x^{2n-1}}\right\} \text{ or, } \tfrac{1}{2}\left\{\frac{\tan.\ \theta}{1^{2n-1}} - \frac{\tan.\ 2\theta}{2^{2n-1}} + \frac{\tan.\ 3\theta}{3^{2n-1}} - \&c.\right\}$$

In order to obtain the coefficients a_1, a_3, ... a_{2n-1}, in the second, we have,

$$f(\varepsilon^t) = \frac{-1}{\sqrt{-1} + \varepsilon^t} = a_0 + a_1 \cdot t + a_2\, t^2 + \&c.$$

Now $f(\varepsilon^t)$ may also be thrown into the form

$$\tfrac{1}{2}\left\{\frac{-1}{\sqrt{-1} + \varepsilon^t} - \frac{-1}{\sqrt{-1} + \varepsilon^{-t}}\right\} + \tfrac{1}{2}\left\{\frac{-1}{\sqrt{-1} + \varepsilon^t} + \frac{-1}{\sqrt{-1} + \varepsilon^{-t}}\right\}$$

which is the same as

$$\tfrac{1}{2}\tan.\left(\frac{t}{\sqrt{-1}}\right) + \tfrac{1}{2}\sqrt{-1} - \tfrac{1}{2}\sec.\left(\frac{t}{\sqrt{-1}}\right)$$

Thus the even values of a_x are given by the developement of sec. $\left(\frac{t}{\sqrt{-1}}\right)$ and the odd by that of tan. $\left(\frac{t}{\sqrt{-1}}\right)$ and hence it is easy to see that

$$a_{2x-1} = \frac{(-1)^{x+1}}{\sqrt{-1}} \cdot \frac{2^{2x-1} \cdot (2^{2x}-1)}{1.2.3 \ldots (2x)} \cdot \mathrm{B}_{2x-1}; \qquad (14)$$

and

$$a_{2x} = \frac{-1}{1.2\ldots(2x).2^{2x+1}}\left\{1^{2x} - \left\{3^{2x} - \frac{2x+1}{1} \cdot 1^{2x}\right\} + \left\{5^{2x} - \&c.\right\} - \&c.\right\}; \quad (15)$$

but a general value of a_x may readily be obtained by the immediate consideration of the function $f(\varepsilon^t)$ itself, as follows:

We know that

$$a_x = \frac{1}{1.2\ldots x} . D^x \frac{-1}{\sqrt{-1} + \left\{ \log^{-1} \right\}} : o$$

Now we have

$$D(f \log^{-1}) : t = \frac{-\varepsilon^t}{(\sqrt{-1} + \varepsilon^t)^2}$$

$$D^2 (f \log^{-1}) : t = \frac{\varepsilon^{2t} - \sqrt{-1} . \varepsilon^t}{(\sqrt{-1} + \varepsilon^t)^3}$$

. . . .

$$D^x (f \log^{-1}) : t = \frac{{}^0A . \varepsilon^{xt} + {}^1A . \varepsilon^{(x-1)t} + \ldots {}^{x-1}A . \varepsilon^t}{(\sqrt{-1} + \varepsilon^t)^{x+1}}$$

In order to determine the numerator of this fraction, we shall adopt the elegant artifice used by LAPLACE * on a similar occasion.

$${}^0A . \varepsilon^{xt} + \ldots . {}^{x-1}A . \varepsilon^t = (\sqrt{-1} + \varepsilon^t)^{x+1} . D^x \left\{ \frac{-1}{\varepsilon^t + \sqrt{-1}} \right\}$$

$$= -(\sqrt{-1} + \varepsilon^t)^{x+1} . D^x \left\{ \varepsilon^{-t} - \sqrt{-1} . \varepsilon^{-2t} - \varepsilon^{-3t} + \sqrt{-1} . \&c. \right\}$$

$$= (-1)^{x+1} . (\sqrt{-1} + \varepsilon^t)^{x+1} . \left\{ 1^x . \varepsilon^{-t} - \sqrt{-1} . 2^x . \varepsilon^{-2t} - 3^x . \varepsilon^{-3t} + \&c. \right\}$$

Now, as this equation is rigorous, and the first member contains only positive powers of ε^t, the negative powers in the second must destroy each other, and may therefore be neglected. Expanding then $(\varepsilon^t + \sqrt{-1})^{x+1}$ in powers of ε^t, multiplying together the two series, and retaining only positive powers of ε^t, we find

* LAPLACE. Mém. de l'Acad. 1779. Sur l'usage du calc. des diff. partielles dans la théorie des suites.

$$^{0}A \,.\, \varepsilon^{xt} + \ldots {}^{x-1}A \,.\, \varepsilon^{t} =$$

$$= (-1)^{x+1} \,.\, \left\{ \begin{array}{l} 1^{x} \,.\, \varepsilon^{xt} - \left\{ 2^{x} - \frac{x+1}{1} \,.\, 1^{x} \right\} \sqrt{-1} \,.\, \varepsilon^{(x-1)t} \\ - \left\{ 3^{x} - \frac{x+1}{1} \,.\, 2^{x} + \frac{(x+1)\,.\,x}{1.2} \,.\, 1^{x} \right\} \,.\, \varepsilon^{(x-2)t} + \&c. \end{array} \right\}$$

And after the substitution of 0 for t, or 1 for ε^{t} and its powers we obtain,

$$a_{x} = \frac{(-1)^{x+1}}{1.2 \ldots\ldots x (1+\sqrt{-1})^{x+1}} \times$$

$$.\left\{ \begin{array}{l} 1^{x} - \left\{ 2^{x} - \frac{x+1}{1}\, 1^{x} \right\} \sqrt{-1} \\ - \left\{ 3^{x} - \frac{x+1}{1} \,.\, 2^{x} + \frac{(x+1)\,.\,x}{1.2} \,.\, 1^{x} \right\} + \&c. \end{array} \right\}; \ldots . (16)$$

The equation (12) will thus take the following form,

$$S\left\{(-1)^{x+1} \,.\, \frac{\tan x\theta}{x^{2n-1}}\right\} = Y_{1}\,\theta + Y_{3}\,\theta^{3} + \ldots Y_{2n-1}\,\theta^{2n-1}; \; (17)$$

where

$$Y_{2x-1} = \frac{2^{2x}\,(2^{2x}-1)\,.\,(2^{2n-2x-1}-1)\,.\,\pi^{2n-2x}}{1.2 \ldots . (2x) \times 1.2 \ldots\ldots (2n-2x)} \,.\, B_{2x-1} \,.\, B_{2n-2x-1}; \; (18)$$

2. Retaining the same form of the function f, and of course the same value of a_{x}, for n write $2n$, and for θ, $\theta \,.\, \sqrt{-1}$, and the first member of (12) becomes

$$\tfrac{1}{2}\left\{\sqrt{-1} \,.\, {}^{2n}L(2) - S\left\{\frac{(-1)^{x+1}}{x^{2n} \,.\, \cos x\theta}\right\}\right\}$$

And the second,

$$a_{0} \,.\, {}^{2n}L(2) - a_{2} \,.\, {}^{2n-2}L(2) \,.\, \theta^{2} + \ldots + (-1)^{n} \,.\, {}^{0}L(2) \,.\, \theta^{2n} \,.\, a_{2n}$$

Now, since $a_{0} = \frac{-1}{1+\sqrt{-1}}$, collecting into one the coefficients of ${}^{2n}L(2)$, viz.: $\sqrt{-1} - 2a_{0}$, we find their sum equal to 1, and of course,

$$S\left\{\frac{(-1)^{x+1}}{x^{2n} \cdot \cos x\theta}\right\} = {}^{2n}L(2) + Y_2 \theta^2 + Y_4 \theta^4 + \ldots Y_{2n} \theta^{2n}; \quad (19)$$

where

$$Y_{2x} = \frac{(-1)^x (2^{2n-2x-1} - 1) \cdot \pi^{2n-2x}}{1.2 \ldots (2x) \times 1.2 \ldots (2n-2x) \cdot 2^{2x}} B_{2n-2x-1} \cdot$$

$$\cdot \left\{ \begin{array}{l} 1^{2x} - \left\{ 3^{2x} - \frac{2x+1}{1} \cdot 1^{2x} \right\} \\ + \left\{ 5^{2x} - \&c \right\} - \&c \end{array} \right\}; \quad (20)$$

If in this equation for θ we write $\pi + \theta$, we shall obtain the sum of the series

$$\frac{1}{1^{2n} \cdot \cos \theta} + \frac{1}{2^{2n} \cdot \cos 2\theta} + \frac{1}{3^{2n} \cdot \cos 3\theta} + \&c; \ldots\ldots (21)$$

And by addition, of

$$\frac{1}{1^{2n} \cdot \cos \theta} + \frac{1}{3^{2n} \cdot \cos 3\theta} + \frac{1}{5^{2n} \cdot \cos 5\theta} + \&c; \ldots\ldots (22)$$

3. Let $f(t) = \frac{\sqrt{-1} - t}{\sqrt{-1} + t}$, and let $2n$ be written for n and $\theta\sqrt{-1}$ for θ; and the first member of (12) becomes

$$\sqrt{-1} \cdot S \left\{ \frac{(-1)^{x+1}}{x^{2n} \cdot \cos x\theta} \right\}$$

Now the developement of $\frac{-1}{\sqrt{-1} + \varepsilon^t}$ being $a_0 + a_1 t + \&c$, that of $f(\varepsilon^t)$ will be

$$(\varepsilon^t - \sqrt{-1}) \{ a_0 + a_1 t + \&c \} =$$

$$= \left\{ (1 - \sqrt{-1}) + \frac{t}{1} + \frac{t^2}{1.2} + \&c \right\} \cdot \left\{ a_0 + a_1 t + \&c \right\}$$

and the coefficient of t^x will be found equal to

$$(1 - \sqrt{-1}) \cdot a_x + \frac{a_{x-1}}{1} + \frac{a_{x-2}}{1.2} + \ldots\ldots \frac{a_0}{1.2 \ldots x}$$

Thus the application of (12) gives the following equation

$$S\left\{\frac{(-1)^{x+1}}{x^{2n}\cdot\cos x\theta}\right\} = {}^{2n}L(2) + Y_2\theta^2 + Y_4\theta^4 + \ldots Y_{2n}\theta^{2n}$$

where

$$Y_{2x} = \frac{(-1)^x\cdot(2^{2n-2x-1}-1)\,\pi^{2n-2x}}{\sqrt{-1}\cdot 1.2\,.\,.\,.\,.\,.\,(2n-2x)} B_{2n-2x-1}\left\{(1-\sqrt{-1})\right.$$

$$\left. a_{2x} + \frac{a_{2x-1}}{1} + \ldots \frac{a_0}{1.2\ldots(2x)}\right\}$$

Which compared with the value of Y_{2x} before found (20) gives the following singular equations,

$$o = a_{2x} + \frac{a_{2x-2}}{1.2} + \frac{a_{2x-4}}{1.2.3.4} + \ldots \frac{a_0 - \frac{1}{2}\sqrt{-1}}{1.2\ldots(2x)}; \ldots\ldots (23)$$

$$a_{2x} = \sqrt{-1}\,.\left\{\frac{a_{2x-1}}{1} + \frac{a_{2x-3}}{1.2.3} + \ldots \frac{a_1}{1.2\ldots(2x-1)}\right\} -$$

$$\frac{1}{2}\cdot\frac{1}{1.2..(2x)}\,. \qquad (24)$$

The latter of these two equations affords the even values of a_x in terms of the odd, and hence we are enabled to express the sum of the series $\frac{1}{1^{2x+1}} - \frac{1}{3^{2x+1}} + \frac{1}{5^{2x+1}} - \&c = {}^{2x+1}C(1)$* by means of the numbers of BERNOUILLI, which EULER appears to have considered as impracticable.† We need hardly re-

* See "Essay on Logarithmic Transcendents," page 51. I should not omit to observe that the equations in p. 69 of that work, expressing the value of the function ${}^nC(x) + (-1)^n\,.\,{}^nC(x^{-1})$ when combined with our equation (10) by making $F(t) = {}^nC(t) = \frac{t}{1n} - \frac{t^3}{3n} + \&c$, afford a series of results, highly interesting, but which the necessary limits of this paper forbid me at present to dilate on.

† "Per hos autem numeros Bernouillianos secans *exprimi non potest, sed requirit* "*alios numeros* qui in summas potestatum reciprocarum imparium ingrediuntur, si "enim ponatur,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c = \alpha\,.\,\frac{\pi}{2^2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \&c = \frac{\beta}{1.2}\,.\,\frac{\pi^3}{2^4},\ \&c.$$

mark that the imaginary form here assumed by a_{2x} is merely apparent.

To proceed. Let the equation (17), multiplied by $d\theta$, be integrated between the limits o and θ, and we find after all reductions.

$$(\sec\theta)^{\left(\frac{1}{1}\right)^{2n}} . (\cos 2\theta)^{\left(\frac{1}{2}\right)^{2n}} . (\sec 3\theta)^{\left(\frac{1}{3}\right)^{2n}} . \&c. = \log^{-1} \left\{ o + Y_1 . \frac{\theta^2}{2} + \ldots\ldots Y_{2n-1} . \frac{\theta^{2n}}{2n} \right\}; \ldots (25).$$

The value of Y_{2x-1} being given by the equation (18)

Again, if we suppose, for the sake of brevity

$$^{2n}L(2) + Y_2 . \theta^2 + \ldots Y_{2n}\,\theta^{2n} = U(\theta)$$

and $^{2n}L(2) . \frac{\theta}{1} + Y_2 . \frac{\theta^3}{3} + \ldots Y_{2n} . \frac{\theta^{2n+1}}{2n+1} = D^{-1}\,U(\theta)$,

we shall obtain, by operating in the same manner on (19) and the equations derived from it, expressing the values of the series (21) and (22).*

" erit $\alpha = 1$, $\beta = 1$, $\gamma = 5$, &c. ex hisque valoribus obtinebitur

$$\sec x = \alpha + \frac{\beta}{1.2} x^2 + \frac{\gamma}{1.2.3.4} x^4 + \&c."$$

Euler. Inst. Calc. Diff. Pars posterior. Cap. VIII. p. 542.

The general value of $^{2x+1}C(1)$ as deduced from our equation (24) in terms of the numbers of Bernouilli is as follows;

$$^{2x+1}C(1) = \left(\frac{\pi}{2}\right)^{2x+1} . \left\{ \frac{2^{2x-1}(2^{2x}-1)}{1.2.3\ldots(2x)} . \frac{B_{2x-1}}{1} - \frac{2^{2x-3}(2^{2x-2}-1)}{1.2\ldots.(2x-2)} . \frac{B_{2x-3}}{1.2.3} + \ldots\ldots\ldots\ldots + (-1)^{x-1} . \frac{2.(2^2-1)}{1.2} . \frac{B_1}{1.2\ldots(2x-1)} + (-1)^x . \frac{1}{2} . \frac{1}{1.2\ldots.(2x)} \right\}$$

* The constant added to complete these integrals is determined by making $\theta = o$ in which case since $\cot\left(\frac{\pi}{4} - x\theta\right) = 1$, their first members vanish when n is greater

$$S\left\{\frac{(-1)^{x+1}}{x^{2n+1}}\log\cot\left(\frac{\pi}{4}-\frac{x\theta}{2}\right)\right\}=D^{-1}U(\theta);\ \ldots\ (26)$$

$$S\left\{\frac{1}{x^{2n+1}}\cdot\log\cot\left(\frac{\pi}{4}-\frac{x\theta}{2}\right)\right\}=D^{-1}U(\pi)-D^{-1}U(\pi+\theta);(27)$$

$$S\left\{\frac{1}{(2x-1)^{2n+1}}\log\cot\left(\frac{\pi}{4}-\frac{(2x-1)\theta}{2}\right)\right\}=$$

$$\frac{D^{-1}\times\left\{U(\pi)+U(\theta)-U(\pi+\theta)\right\}}{2};\ \ldots\ldots\ldots\ (28).$$

In the last of these equations, if we write $-\left(\frac{\pi}{2}+2\theta\right)$ for θ, we obtain the following equation, corresponding to (25)

$$(\tan\theta)^{\left(\frac{1}{1}\right)^{2n+1}}\cdot(\cot 3\theta)^{\left(\frac{1}{3}\right)^{2n+1}}\cdot(\tan 5\theta)^{\left(\frac{1}{5}\right)^{2n+1}}\ \&c=$$

$$\log^{-1}\left\{\frac{D^{-1}U(\pi)+D^{-1}U\left(-\frac{\pi}{2}-2\theta\right)-D^{-1}U\left(\frac{\pi}{2}-2\theta\right)}{2}\right\};\ .\ (29).$$

These are but a few of the very singular results which may be deduced from our equation (12); but I shall forbear to extend this paper to an unnecessary length by any farther applications of it.

than unity; but when $n=o$, the series $S\left\{\frac{1}{x^{2n+1}}\right\}$ and $S\left\{\frac{1}{(2x-1)^{2n+1}}\right\}$ becoming infinite, they take the forms $\infty\times\log(1)$ which is altogether vague and inconclusive. Our equations (27) and (28), (29), then, are defective in this case, and we can only conclude that the function

$$(\tan\theta)^{\frac{1}{1}}\cdot(\cot 3\theta)^{\frac{1}{3}}\cdot\&c.$$

is independent on θ, or constant. There seems reason to conclude from other principles however that this constant is $\varepsilon^{-\frac{\pi^2}{8}}$, or more generally, $\varepsilon^{(2i+1)\frac{\pi^2}{8}}$ i being any integer, positive or negative.

III. *On Functional Equations.*

The determination of functions from given conditions is a point of such importance, not only in the partial differential calculus, but also in a variety of other branches, that it has occupied the attention of the most eminent Analysts, and it must be confessed, not without considerable success. Their researches, however, have hitherto extended no farther than to such conditions as involve only the unknown function, φ without any of its superior or inferior orders, φ^2, φ^3, . . . &c, φ^{-1}, &c. It is to equations of this latter kind, therefore, that we now propose to direct our attention.

The successive orders of any function $f(x)$ may be produced, either by actually writing $f(x)$ for x in the expression of $f(x)$, in which case the general value of $f^z(x)$ must be concluded from induction; or more elegantly by the following method.

Assume $f^z(x) = u_z$ and we have $f^{z+1}(x) = u_{z+1}$

$$o = u_{z+1} - f(u_z)$$

an equation of differences whose integral will be of the form

$$u_z = \mathrm{F}(z, \mathrm{C})$$

C being an arbitrary quantity independent on z. Let $z = o$, and we have

$$\mathrm{F}(o, \mathrm{C}) = u_o = f^o(x) = x$$

an equation which gives C in functions of x.

For example; let $f(x) = 2x^2 - 1$, and we have

$$o = u_{z+1} - 2u_z^2 + 1$$

and integrating,

$$u_x = \tfrac{1}{2}\left\{C^{2^x} + C^{-2^x}\right\}$$

Now, if we make $u_0 = x$, we get

$$C = x + \sqrt{x^2 - 1}$$

and consequently,

$$f^x(x) = \tfrac{1}{2}\left\{\left(x + \sqrt{x^2 - 1}\right)^{2^x} + \left(x - \sqrt{x^2 - 1}\right)^{2^x}\right\}.$$

If we suppose $\varphi\,(x, y)$ to denote any function of x and y, and conceive this expression substituted for x, as follows;

$$\varphi\left\{\varphi\,(x, y), y\right\},$$

we shall have the second *partial function*, taken with respect to x, which we may denote thus, $\varphi^{2,1}(x, y)$. If we repeat, or conceive repeated, this operation m times, we shall have the mth partial function with respect to x:

$$\varphi^{m,1}(x, y) = \varphi\left\{\varphi^{m-1,1}(x, y)\,, y\right\}.$$

If the mth partial function with respect to x be in like manner successively substituted n times for y in the expression $\varphi\,(x, y)$, we shall obtain a result,

$$\varphi^{m,n}(x, y) = \varphi\left\{x, \varphi^{m,n-1}(x, y\right\},$$

and so on for more variables, z, w, &c. —. An equation containing any number of the successive orders $\varphi^0(x) = x$, $\varphi\,(x)$, ... $\varphi^n(x)$, of a function φ, and from which φ is to be determined, is called a functional equation of the nth order, in φ. Thus the equation

$$o = \varphi^2(x) - (1 + b)\,.\,\varphi\,(x) + bx$$

is a functional equation of the second order, and is satisfied by the following

$$\varphi\,(x) = a + bx.$$

An equation between any number of the partial functions $\phi^{m,n,\&c}(x, y, z, \&c)$ for determining the form of $\phi(x, y, z, \&c)$ is called an equation of partial functions, and its order may be denoted in the same manner. Thus

$$o = \phi^{2,1}(x,y) + \phi^{1,2}(x,y) - (a+b+1).\phi(x,y) + c$$

is an equation of partial functions of the second order with two variables, and is satisfied by the equation

$$\phi(x,y) = ax + by + c$$

Let $\phi(x)$ be a function of x, and a certain number of constants $a, b, c, \ldots$ And from this expression conceive $\phi^{\alpha}(x)$, $\phi^{\beta}(x)$ &c to be successively formed which will be functions also of a, b, &c. If the number of these constants be n, we may thus produce $n+1$ such functions of them, which will be respectively equal to the several orders of ϕ which they represent. Thus we have $n+1$ equations involving the n quantities a, b, c, &c which may therefore be eliminated, and the resulting equation between x, $\phi^{\alpha}(x)$, $\phi^{\beta}(x)$ &c will therefore be independent on them. As far then as regards this equation they are arbitrary, so that in reascending from a functional equation which contains $n+1$ different orders of ϕ (not including $\phi^{0}(x)$ or x) n arbitrary constants must be introduced. The reasoning here made use of is sufficiently plausible, and in fact, no other than has been adduced in demonstration of well known and important truths. The conclusion too, to the extent of its literal meaning, is correct. But we have here to notice a paradox of a very singular nature, viz: that even in the simplest cases imaginable (such as $\phi^{2}(x) = x$) the general expression for $\phi(x)$ may contain, not one or two, but an unlimited number of arbitrary constants, nay

even one or more arbitrary functions. A nearer attention to every step of the above reasoning will explain this paradox. But what has been said will serve to make us cautious in trusting implicitly to all its other applications.

Problem I. To determine $\varphi(x)$ from the equation $\varphi^2(x) = x$. Assume z a function of x, and u a functional characteristic, which shall satisfy the following conditions

$$x = u_z, \; \varphi(x) = u_{z+1}.$$

From these, we obtain

$$\varphi(x), \text{ that is, } \varphi(u_z) \text{ or } (\varphi u)_z = u_{z+1}; \; \ldots\ldots\ldots (a)$$

$$\text{and } \varphi^2(x) \text{ or } \varphi\left\{\varphi(x)\right\} = \underline{(\varphi u)_{z+1} = x = u_z}; \; \ldots\ldots (b)$$

$$\text{and, subtracting, } (\varphi u)_{z+1} - (\varphi u)_z = -(u_{z+1} - u_z)$$

$$\text{that is, } \Delta\left\{(\varphi u)_z + u_z\right\} = o; \; \ldots\ldots\ldots\ldots (c)$$

and integrating,

$$o = (\varphi u)_z + u_z + \mathrm{C}.$$

Now by cross-multiplication of the equations (a) and (b) we find,

$$u_{z+1} \cdot (\varphi u)_{z+1} = u_z \cdot (\varphi u)_z.$$

Thus the function $u_z \cdot (\varphi u)_z$ does not vary when z changes to $z+1$, and of course must be considered as constant in the integration of (c). C therefore may be any function of $u_z \cdot (\varphi u)^z$, and thus our equation becomes

$$o = u_z + (\varphi u)_z + f\left\{u_z \cdot \varphi(u_z)\right\}$$

$$\text{or } o = x + \varphi(x) + f\left\{x \cdot \varphi(x)\right\}$$

an equation from which $\varphi(x)$ may be obtained for any assigned form of the function f. Thus if $f(x) = a + bx$,

$$o = a + x + (1 + bx) \cdot \varphi(x)$$

$$\text{and } \varphi(x) = -\frac{a+x}{1+bx}$$

which satisfies the condition proposed; and by giving f other forms, we should obtain other values of $\varphi(x)$.

The subsidiary function z, and the characteristic u are not then necessary to be *known* but as a matter of curiosity. They may however be found when φ is determined, by the resolution of the equation of differences $\varphi(u_z) = u_{z+1}$ which gives the form of the function u_z in z, and z is given by the equation $x = u_z$, or $z = u^{-1}(x)$.

Aliter. Assume as before, $x = u_z$, $\varphi(x) = u_{z+1}$ then we have

$$\varphi(u_{z+1}) = u_z; \dots\dots\dots\dots\dots (d).$$

Now, $x = u_z$ therefore $\varphi(x) = \varphi(u_z)$, that is, $u_{z+1} = \varphi(u_z)$ and for z writing $z - 1$, $u_z = \varphi u_{z-1}$ which being substituted in (d) gives

$$\varphi(u_{z+1}) = \varphi(u_{z-1}).$$

Now this is a perfect function φ on both sides, and of course, taking the inverse function φ^{-1} on both sides

$$u_{z+1} = u_{z-1}$$

whence,

$$u_z = \mathrm{C}\left\{\cos 2\pi z\right\} + (-1)^z \cdot \mathrm{C}'\left\{\cos 2\pi z\right\}$$

C and C′ being two arbitrary functional characteristics. Now $u_z = x$, and consequently

$$x = \mathrm{C}\left\{\cos 2\pi z\right\} + (-1)^z \cdot \mathrm{C}'\left\{\cos 2\pi z\right\}.$$

From this conceive z found in functions of x, and call it $\mathrm{Z}(x)$ then,

$$u_{z+1} = \varphi(x) = C\left\{\cos 2\pi Z(x)\right\} - (-1)^{Z(x)} . C'\left\{\cos 2\pi Z(x)\right\}.$$

This method applies also to the more general equation $\varphi^2(x) = f(x)$, by the substitutions $f(x) = u_z$, $\varphi(x) = u_{z+1}$, but, owing to the transcendental equations it introduces, must be regarded as totally ineffectual and useless.

Prob. II. Given $\varphi^n(x) = f(x)$. Required at least one satisfactory value of $\varphi(x)$.

Let the general expression of $f^z(x)$ in functions of z and x found according to the method above explained be $F\left\{z, x\right\}$: we have then $\varphi^n = f$, and $\varphi = f^{\frac{1}{n}}$, that is

$$\varphi(x) = f^{\frac{1}{n}}(x) = F\left\{\frac{1}{n}, x\right\}.$$

Ex. 1. Let $f(x) = 2x^2 - 1$, or $\varphi^2(x) = 2x^2 - 1$, and we find

$$f^z(x) = \tfrac{1}{2}\left\{\left(x + \sqrt{x^2-1}\right)^{2^z} + \left(x - \sqrt{x^2-1}\right)^{2^z}\right\}$$

and of course

$$f^{\frac{1}{n}}(x) = \varphi(x) = \tfrac{1}{2}\left\{\left(x + \sqrt{x^2-1}\right)^{\sqrt[n]{2}} + \left(x - \sqrt{x^2-1}\right)^{\sqrt[n]{2}}\right\}.$$

We may here observe that *any one of the n values of* $\sqrt[n]{2}$ will equally afford a satisfactory value of $\varphi(x)$.

Ex. 2. Let $\varphi^n(x) = \frac{\alpha + \beta x}{\gamma + \delta x} = f(x)$

Assume $\varpi = \frac{\beta - \gamma}{2}$, $\lambda = \frac{\beta + \gamma}{2}$, $\mu = \left\{\varpi^2 + \alpha\delta\right\}^{\frac{1}{2}}$, $\nu = \frac{\lambda - \mu}{\lambda + \mu}$ and we shall find

$$f^{\frac{1}{n}}(x) \text{ or } \varphi(x) = \frac{\left\{\alpha + (\varpi - \mu)x\right\} . \nu^{\frac{1}{n}} - \left\{\alpha + (\varpi + \mu)x\right\}}{\left\{\delta x - (\varpi + \mu)\right\} . \nu^{\frac{1}{n}} - \left\{\delta x + (\varpi - \mu)\right\}}$$

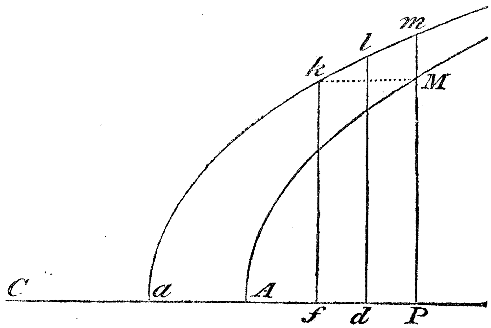
where any of the n values of $\nu^{\frac{1}{n}}$ may be taken, and thus as

many different values of $\varphi(x)$ be obtained. This example depends on the integration of an equation of differences of the form

$$o = u_{z+1} \, . \, u_z + \mathrm{A} \, . \, u_{z+1} + \mathrm{B} \, . \, u_z + \mathrm{C}$$

a particular case of which had been previously integrated by LAPLACE in the Journal de l'Ecole Polytechnique.

Ex. 3. To take an instance of the application of these equations to geometrical problems, let AM be an hyperbola whose axis is CP and centre C, and let it be required to find a curve am such that drawing the ordinate PMm, making $Cd = Pm$ and again erecting dl parallel to PM, if this be repeated n times the last ordinate fk shall be equal to PM. Let $y = \varphi(x)$ be the equation of am and $y^2 = (1 - e^2)(a^2 - x^2)$ that of AM, then $dl = \varphi(Cd) = \varphi(Pm) = \varphi^2(x)$ and in like manner, $fk = \varphi^n(x) = PM$, that is, $\varphi^n(x) = f(x) = \sqrt{(1 - e^2)(a^2 - x^2)}$; consequently,



$$f^{\frac{1}{n}}(x) = \varphi(x) = \left\{(e^2 - 1)^{\frac{1}{n}} \, . \, x^2 - \frac{e^2 - 1}{e^2 - 2}\left\{(e^2 - 1)^{\frac{1}{n}} - 1\right\} . \, a^2\right\}^{\frac{1}{2}}.$$

Thus we see, that am is also an hyperbola, whose centre is C, and calling a' and e' its semiaxis and excentricity, we have

$$e' = \sqrt{(e^2 - 1)^{\frac{1}{n}} + 1}, \text{ and } a' = a \, . \left\{\frac{e^2 - 1}{e'^2 - 1} \, . \, \frac{e'^2 - 2}{e^2 - 2}\right\}^{\frac{1}{2}}.$$

If AM be a right angled hyperbola, or $e = \sqrt{2}$, we shall have $e' = \sqrt{2}$ and $a' = \frac{a}{\sqrt{n}}$; that is, am is also a right angled hyperbola, having its axis $\frac{1}{\sqrt{n}}$ part of that of AM. If e

be < 1, and n odd, we shall have e' also < 1. Thus if the curve AM be an ellipse, am is also a concentric ellipse. The equations of *Ex.* 2. geometrically expressed afford a property of the hyperbola something similar.

IV. *On differential Equations of the first degree.*

Any equation of this species may be reduced to the form

$$o = u + {}^{1}\mathrm{A} \,.\, \mathrm{D}u + {}^{2}\mathrm{A} \,.\, \mathrm{D}^{2}u + \ldots . {}^{n}\mathrm{A} \,.\, \mathrm{D}^{n}u + \mathrm{X}\,; \quad \ldots (1)$$

u being the unknown, and ${}^{1}\mathrm{A}, \ldots . {}^{n}\mathrm{A}$, X, known functions of x. To integrate it, assume the following equations

$$\left.\begin{array}{l} u + {}^{1}\alpha \mathrm{D}u = u^{(1)} \\ u^{(1)} + {}^{2}\alpha \mathrm{D}u^{(1)} = u^{(2)} \\ \cdots\cdots\cdots\cdots \\ u^{(n-2)} + {}^{n-1}\alpha \mathrm{D}u^{(n-2)} = u^{(n-1)} \\ u^{(n-1)} + {}^{n}\alpha \mathrm{D}u^{(n-1)} + \mathrm{X} = u^{(n)} = o \end{array}\right\}; \quad \ldots\ldots (2)$$

From these, eliminating successively $u^{(1)}, \ldots u^{(n-1)}$, we obtain

$$u^{(1)} = u + {}^{1}\alpha \,.\, \mathrm{D}u$$

$$u^{(2)} = u + ({}^{1}\alpha + {}^{2}\alpha + {}^{2}\alpha \mathrm{D}^{1}\alpha) \,.\, \mathrm{D}u + {}^{1}\alpha \,.\, {}^{2}\alpha \,.\, \mathrm{D}^{2}u$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$o = u^{(n)} = u + \left\{ {}^{1}\alpha + {}^{2}\alpha (1 + \mathrm{D}^{1}\alpha) + \&c \right\} . \mathrm{D}u + \ldots .$$

$${}^{1}\alpha \,.\, {}^{2}\alpha \ldots {}^{n}\alpha \,.\, \mathrm{D}^{n}u + \mathrm{X}\,; \quad \ldots\ldots (3)$$

The comparison of the coefficients of this equation with ${}^{1}\mathrm{A}, \ldots {}^{n}\mathrm{A}$, gives n equations for determining ${}^{1}\alpha, \ldots {}^{n}\alpha$, into which X does not enter. Consequently these functions are independent on X, and therefore, the same as if $\mathrm{X} = o$. Now the successive integration of (2) gives

$$u = \left\{\log^{-1}\int\frac{-dx}{{}^{1}\alpha}\right\}.\int\frac{\log^{-1}\int\left(\frac{dx}{{}^{1}\alpha}-\frac{dx}{{}^{2}\alpha}\right)}{{}^{1}\alpha}\int\ \ldots\ldots\ldots$$

$$\int\frac{\log^{-1}\int\left(\frac{dx}{{}^{n-1}\alpha}-\frac{dx}{{}^{n}\alpha}\right)}{{}^{n-1}\alpha}\int\frac{-Xdx^{n}}{{}^{n}\alpha}\log^{-1}\int\frac{dx}{{}^{n}\alpha}; \quad \ldots \quad (4)$$

which by writing o for X, and adding a constant at each integration becomes,

$$u = {}^{1}C.\log^{-1}\int\frac{-dx}{{}^{1}\alpha} + {}^{2}C.\left\{\log^{-1}\int\frac{-dx}{{}^{1}\alpha}\right\}.\int\frac{\log^{-1}\int\left(\frac{dx}{{}^{1}\alpha}-\frac{dx}{{}^{2}\alpha}\right)}{{}^{1}\alpha}.dx + \&c. \ (5)$$

Now, if ${}^{(1)}u$, ${}^{(2)}u$, ... ${}^{(n)}u$ be the particular integrals of

$$o = u + {}^{1}A.Du + \ldots {}^{n}A.D^{n}u; \quad \ldots\ldots\ldots \quad (6)$$

we shall have, when $X = o$,

$$u = {}^{1}C.{}^{(1)}u + {}^{2}C.{}^{(2)}u + {}^{3}C.{}^{(3)}u + \&c.$$

And comparing this with the expression (5), we find

$$\frac{-1}{{}^{1}\alpha} = D\log{}^{(1)}u;$$

$$\frac{-1}{{}^{2}\alpha} = D\log\left\{{}^{1}\alpha.{}^{(1)}u.D\frac{{}^{(2)}u}{{}^{(1)}u}\right\};$$

$$\frac{-1}{{}^{3}\alpha} = D\log\left\{{}^{1}\alpha.{}^{2}\alpha.{}^{(1)}u.D\frac{{}^{(2)}u}{{}^{(1)}u}.D\frac{D\frac{{}^{(3)}u}{{}^{(1)}u}}{D\frac{{}^{(2)}u}{{}^{(1)}u}}\right\};$$

&c = &c.

Suppose now that by any means whatever we can discover $n-1$ particular integrals ${}^{(1)}u$, ... ${}^{(n-1)}u$, of (6), the original equation deprived of its last term: then by the help of the first $n-1$ of these equations, the values of ${}^{1}\alpha$, ... ${}^{n-1}\alpha$, are given,

and from these, ${}^{n}\alpha$ may be derived by considering that the comparison of the equations (1) and (3) gives

$$ {}^{1}\alpha \,.\, {}^{2}\alpha \ldots\ldots {}^{n-1}\alpha \,.\, {}^{n}\alpha = {}^{n}\mathrm{A}, \text{ or } {}^{n}\alpha = {}^{n}\mathrm{A} \,.\, ({}^{1}\alpha \ldots {}^{n-1}\alpha)^{-1} $$

Having obtained ${}^{1}\alpha, \ldots\ldots {}^{n}\alpha$, nothing more is requisite for obtaining a complete integral of (1), than to substitute their values in equation (4).

The method here delivered of obtaining the known theorems respecting the equation

$$ o = u + {}^{1}\mathrm{A}\mathrm{D}u + \ldots {}^{n}\mathrm{A} \,.\, \mathrm{D}^{n}u + \mathrm{X} $$

appears to have the advantage in point of conciseness over any I have hitherto met with; a sufficient apology for the revival of a subject whose theory, and whose difficulties have been so long and completely understood.

In the case when $\mathrm{X} = o$ and ${}^{1}\mathrm{A}, \ldots {}^{n}\mathrm{A}$ are constant, the method of separating the symbols of operation from those of quantity, may be introduced with great elegance.

Let p, q, r, &c be the roots of

$$ o = \mathrm{D}^{n} + \frac{{}^{n-1}\mathrm{A}}{{}^{n}\mathrm{A}} \mathrm{D}^{n-1} + \ldots + \frac{1}{{}^{n}\mathrm{A}} $$

and the equation (1) becomes

$$ o = (\mathrm{D} - p)(\mathrm{D} - q) \ldots \&c : u $$

which is satisfied by either of the equations

$$ o = (\mathrm{D} - p) : u, \qquad o = (\mathrm{D} - q) : u, \text{ \&c. or,} $$

$$ \mathrm{D}u = pu, \qquad \mathrm{D}u = qu, \text{ \&c.} $$

Now these equations integrated give the following

$$ u = \varepsilon^{px}, \; u = \varepsilon^{qx}, \; u = \varepsilon^{rx}, \text{ \&c.} $$

which are the particular integrals of the proposed, and of course its complete integral will be

$$ u = {}^{1}\mathrm{C} \,.\, \varepsilon^{px} + {}^{2}\mathrm{C} \,.\, \varepsilon^{qx} + {}^{3}\mathrm{C} \,.\, \varepsilon^{rx} + \text{\&c.} $$

If there be m roots equal to p, we have

$$(D - p)^m : u = o, \text{ or } (D - p)^m : u \times \varepsilon^{-px} = o.$$

$$\text{Now, } (D - p)^m : u \,.\, \varepsilon^{-px} = D^m \left\{ u \,.\, \varepsilon^{-px} \right\} = o.$$

Therefore, integrating m times

$$u \,.\, \varepsilon^{-px} = {}^{0}C + {}^{1}C \,.\, x + \ldots\ldots {}^{m-1}C \,.\, x^{m-1}$$

and $$u = \left\{ {}^{0}C + {}^{1}Cx + \ldots\ldots {}^{m-1}C \,.\, x^{m-1} \right\} \varepsilon^{px}$$

which is the part of the integral arising from the equal roots p.

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